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ZERO-SUM SUBSEQUENCES IN ABELIAN NON-CYCLIC GROUPS

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ABSTRACT

Let G be a finite abelian group, $G \notin \{Z_n, Z_2 \oplus Z_{2n}\}$. Then every sequence $A = \{g_1, \ldots, g_t\}$ of $t = \frac{4|G|}{3} + 1$ elements from G contains a subsequence $B \subset A$, $|B| = |G|$ such that $\sum_{g_i \in B} g_i = 0$ (in G). This bound, which is best possible, extends recent results of [1] and [22] concerning the celebrated theorem of Erdös–Ginzburg–Ziv [21].

1. Introduction

Thirty years ago, Erdös, Ginzburg and Ziv proved the following celebrated theorem.

THEOREM A ([21]): *Let $m \geq k \geq 2$ be positive integers such that $k|m$, and let $A = \{a_1, a_2, \ldots, a_{m+k-1}\}$ be a sequence of integers. Then there exists $I \subset \{1, 2, \ldots, m+k-1\}$, $|I| = m$, such that $\sum_{i \in I} a_i \equiv 0$ (mod k).*

This theorem is the starting point of many new results in the evolving area called Zero-sum theory. We refer the reader to the references.

Recently efforts have been made to make precise the Erdös–Ginzburg–Ziv theorem culminating in the following results of Alon–Bialostocki–Caro and Flores-Ordaz.

THEOREM B ([1], [22]): *Let $A = \{a_1, \ldots, a_{m+k-2}\}$ be a sequence of integers that violates the conclusion of Theorem A. Then there are two elements $a, b \in Z_k$ (the cyclic group mod k) such that*

(1) $\gcd(a-b, k) = 1$ *(namely $a - b$ is a generator of Z_k),*

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(2) $A = A_1 \cup A_2$, $|A_1| \equiv |A_2| \equiv -1$ *(mod k) and* $a_i \in A_1$ *implies* $a_i \equiv a$ *(mod k), while* $a_i \in A_2$ *implies* $a_i \equiv b$ *(mod k).*

THEOREM C ([1], [22]): *Let G be an abelian non-cyclic group of order n, $n|m$, and let $A = \{g_1, \ldots, g_{n+m-2}\}$ be a sequence of $n+m-2$ elements from G. Then there exists $S \subset A$, $|S| = m$ such that $\sum_{g_i \in S} g_i = 0$ (in G).*

A stronger result proved in [1] is:

THEOREM D ([1]): *Let G be a finite abelian non-cyclic group of order n, and let $A = \{g_1, \ldots, g_t\}$, $t = 3n/2$, be a sequence of elements from G. Then*

(1) *There exists $B \subset A$, $|B| = n$ such that $\sum_{g_i \in B} g_i = 0$ (in G).*

(2) *The bound $t = 3n/2$ is best possible and is realized only by groups of the form $G = Z_2 \oplus Z_{2m}$.*

Our main result is the following:

THEOREM 1: *Let G be a finite abelian group, $G \notin \{Z_n, Z_2 \oplus Z_{2n}\}$ and let $A = \{g_1, \ldots, g_t\}$, $t = 4|G|/3 + 1$, be a sequence of elements from G. Then*

(1) *There exists $B \subset A$, $|B| = |G|$, such that $\sum_{g_i \in B} g_i = 0$ (in G).*

(2) *The bound $t = 4|G|/3 + 1$ is best possible and is realized only by groups of the form $G = Z_3 \oplus Z_{3n}$.*

We assume from now on that H is a finite abelian group. The Davenport's constant of G, denoted by $D(G)$, is the smallest integer t such that every sequence of t members of G contains a subsequence whose members sum to 0 (in G). The zero-sum constant of G, denoted by $\mathrm{ZS}(G)$, is the smallest integer t such that any sequence of t members of G contains a subsequence of cardinality $|G|$, that sum to 0 (in G).

Thus $\mathrm{ZS}(Z_n) = 2n - 1$ by Theorem A, while it is easy to see that $D(Z_n) = n$.

2. Preliminary results

The proof of Theorem 1 is quite lengthy and requires some preliminary results, some of which were already explored in [1]. It becomes clear in the course of the proof of Theorem 1 that our proof inevitably contains the proof of Theorem D mentioned in the introduction. Due to the length of the whole proof, it is more comfortable to split it into several, almost independent sections.

The following theorem of Olson [26] is an important tool.

THEOREM E ([26]): *Let G be an abelian p-group (p prime) of the form $G = Z_{p^{\alpha_1}} \oplus \cdots \oplus Z_{p^{\alpha_k}}$. Then $D(G) = 1 + \sum_{i=1}^{k}(p^{\alpha_i} - 1)$.*

Using Olson's theorem we can compute the zero-sum constant for abelian p-groups.

THEOREM 2: *Let G be an abelian p-group of the form $G = Z_{p^{\alpha_1}} \oplus \cdots \oplus Z_{p^{\alpha_k}}$. Then*

$$\mathrm{ZS}(G) = 1 + \sum_{i=1}^{k}(p^{\alpha_i} - 1) + \left(p^{\sum_{i=1}^{k}\alpha_i} - 1\right) = |G| + D(G) - 1.$$

Proof: Let $H = G \oplus Z_{|G|}$, then H is also a p-group and, by Olson's theorem,

$$D(H) = 1 + \sum_{i=1}^{k}(p^{\alpha_i} - 1) + \left(p^{\sum_{i=1}^{k}\alpha_i} - 1\right) = |G| + D(G) - 1.$$

Now let $A = \{g_1, \ldots, g_t\}$ be a sequence of $t = D(H)$ elements from G. Consider the related sequence $B = \{h_1, \ldots, h_t\}$, $t = D(H)$ where $h_i = (g_i, 1) \in H$. Then by the definition of $D(H)$ there exists a subsequence of B that sums to 0 in H. But $D(H) = |G| + D(G) - 1 < 2|G|$, hence the second coordinate of the h_i's (1 in $Z_{|G|}$) forces that the number of summands is exactly $|G|$, thus $\mathrm{ZS}(G) \leq D(H) = |G| + D(G) - 1$. On the other hand, let $e_i \in G$ be the vector (element of G) with k coordinates, whose i-th coordinate is 1 and otherwise is 0. Consider the following sequence A of members of G:

Take $(p^{\alpha_i} - 1)$ copies of e_i for $i = 1, \ldots, k$, and $|G| - 1$ copies of the zero-vector $\mathbf{0} = (0, 0, \ldots, 0)$. Clearly $|A| = |G| + D(G) - 2$ but A contains no subsequence of $|G|$ members that sums to 0. Thus $\mathrm{ZS}(G) = |G| + D(G) - 1$. ∎

Theorem 2 suggests the following conjecture:

CONJECTURE 0: *Let G be a finite abelian group. Then $\mathrm{ZS}(G) = |G| + D(G) - 1$. It is easy to see that $\mathrm{ZS}(G) \geq |G| + D(G) - 1$ always holds.*

Another useful result is:

THEOREM 3: *Let G be an abelian group written as $G = A \oplus H$, then $\mathrm{ZS}(G) \leq (\mathrm{ZS}(A) - 1)|H| + \mathrm{ZS}(H)$.*

Proof: This bound allows us to choose $\mathrm{ZS}(A)$ blocks, each of $|H|$ elements, in each of which the sum of the second coordinate (H) is 0 in H. Hence we must have (by the definition of ZS(A)) exactly $|A|$ blocks of $|H|$ elements whose sum in the first coordinate is also 0 in A, hence 0 in G. ∎

An interesting conjecture (see [3]) is:

CONJECTURE 1: *Let $A = \{a_1, \ldots, a_{4n-3}\}$ be a sequence of elements in $Z_n \oplus Z_n$. Then there exists $I \subset \{1, \ldots, 4n-3\}$, $|I| = n$ such that $\sum_{i \in I} a_i = 0$ in $Z_n \oplus Z_n$.*

The bound $4n-3$ is known to hold for $2 \leq n \leq 6$ (by messy calculations) and in [3] an upper bound $6n-7$ is proved for all n and $5n-2$ is an upper bound for n sufficiently large. We shall use only the $4n-3$ bound for $2 \leq n \leq 5$.

The last tool we need is the Baker–Schmidt theorem [5].

THEOREM F ([5]): *Let q be a prime power and let $h_i(X) = h_i(x_1, \ldots, x_t) \in Z[x_1, \ldots, x_t]$, $i = 1, \ldots, n$ be a family of polynomials satisfying:*

$$h_1(0) \equiv \cdots \equiv h_n(0) \equiv 0 \pmod q, \quad \text{and also} \quad t > \left(\sum_{i=1}^{n} \deg h_i(x)\right)(q-1).$$

Then there exists an $0 \neq \alpha \in \{0,1\}^t$ such that $h_1(\alpha) \equiv \cdots \equiv h_n(\alpha) \equiv 0 \pmod q$.

3. Some exact computation of ZS(G)

Our main result in this section is:

THEOREM 4:

(1) $\mathrm{ZS}(Z_2 \oplus Z_{2m}) = 6m$,
(2) $\mathrm{ZS}(Z_3 \oplus Z_{3m}) = 12m + 1$,
(3) $\mathrm{ZS}(Z_4 \oplus Z_{4m}) = 20m + 2$,
(4) $\mathrm{ZS}(Z_5 \oplus Z_{5m}) = 30m + 3$.

Proof: Since $\mathrm{ZS}(Z_2 \oplus Z_{2m}) = 6m$ is proved in [1] and is slightly simpler than the other cases we shall prove the case $\mathrm{ZS}(Z_3 \oplus Z_{3m}) = 12m + 1$. So let $e_1 = (1,0)$, $e_2 = (0,1)$, $e_3 = (1,1)$ be members of $Z_3 \oplus Z_{3m}$.

Take $3m-1$ copies of e_1, $9m-1$ copies of e_2 and two of e_3 to get a sequence of $12m$ elements in $Z_3 \oplus Z_{3m}$ without a zero-sum subsequence of cardinality $9m$. Hence $\mathrm{ZS}(Z_3 \oplus Z_{3m}) \geq 12m + 1$. In fact the lower bound in all the cases cited in Theorem 4 follows directly from $\mathrm{ZS}(G) \geq |G| + D(G) - 1$.

To show the converse inequality, which is much harder, let

$$A = \{g_1, \ldots, g_{12m+1}\}$$

be a sequence of elements in $Z_3 \oplus Z_{3m}$.

We need the following facts.

FACT 1: Let $a_1, \dots, a_9$ be 9 elements in $Z_3 \oplus Z_3$. There exist three of them whose sum is 0 in $Z_3 \oplus Z_3$.

This is a special case of Conjecture 1 that any sequence of $4n-3$ elements in $Z_n \oplus Z_n$ contains a zero-sum subsequence of cardinality n.

This has been checked for $2 \le n \le 6$ and is easy for $n = 3$.

FACT 2: Let $a_1, \dots, a_7$ be 7 elements in $Z_3 \oplus Z_3$. There exist either three or six of them whose sum in 0 in $Z_3 \oplus Z_3$.

This follows from Theorem F [5] in the following way:

Write $a_i = (b_i, c_i)$, $b_i, c_i \in Z_3$. Consider the following polynomial equations.

$$f_1(X) = \sum_{i=1}^{7} b_i x_i \equiv 0 \pmod 3,$$

$$f_2(X) = \sum_{i=1}^{7} c_i x_i \equiv 0 \pmod 3,$$

$$f_3(X) = \sum_{i=1}^{7} x_i \equiv 0 \pmod 3.$$

Since $7 > \left(\sum_{i=1}^{3} \deg f_i(x)\right)(3-1) = 6$ and $x_i \equiv 0$, $i = 1, \dots, 7$ is a solution, then by the Baker–Schmidt theorem there is another solution with $x_i \in \langle 0, 1\rangle$, $i = 1, \dots, 7$. But $f_3(x)$ implies that we have chosen either 3 or 6 members.

Let us return to the proof.

Consider the members of A over $Z_3 \oplus Z_3$ first. By the last two observations and since $|A| = 12m + 1$ we must have either $4m-1$ triples, say $A_1, \dots, A_{4m-1}$, $|A_i| = 3$, such that each triple sums to 0 in $Z_3 \oplus Z_3$, or $4m-2$ such triples, say $A_1, \dots, A_{4m-2}$, $|A_i| = 3$, and a 6-tuple B such that they sum each to 0 in $Z_3 \oplus Z_3$.

We now concentrate on the sums of the second coordinates in the A_i's and B. For each $1 \le i \le 4m-2$ write

$$d_i = \frac{1}{3}\{\text{the sum of the second coordinates of the members of } A_i\}$$

and d_{4m-1} is this sum for A_{4m-1} respectively B.

CASE 1: Suppose we have $4m-1$ zero-sum triples and consider

$$D = \{d_1, d_2, \ldots, d_{4m-1}\}.$$

By the Erdös–Ginzburg–Ziv theorem there exists $I \subset \{1, \ldots, 4m-1\}$, $|I| = 3m$, such that $\sum_{i \in I} d_i \equiv 0 \pmod{m}$. Hence $\sum_{g_j \in A_i} g_j = 0$ in $Z_3 \oplus Z_{3m}$ forming a zero-sum subsequence of cardinality $I \cdot |A_i| = 9m$ as needed.

CASE 2: Suppose we have $4m-2$ zero-sum triples and the 6-tuples B., Consider $D = \{d_1, \ldots, d_{4m-2}\}$ and d_{4m-1}. By Theorem B we may repeat the argument of case 1 unless there exist $a, b \in Z_m$ $\gcd(a-b, m) = 1$ and either (w.l.o.g.) $d_1 = \cdots = d_{3m-1} = a$ (in Z_m); $d_{3m} = \cdots = d_{4m-2} = b$ (in Z_m) or $d_1 = \cdots = d_{2m-1} = a$ (in Z_m) and $d_{2m} = \cdots = d_{4m-2} = b$ (in Z_m).

Suppose $d_{4m-1} \equiv j \pmod{m}$. We have to find $3m-2$ of the d_i's whose sum with $d_{4m-1} \equiv 0 \pmod{m}$. So we have either

(I) $j + bx + (3m-2-x)a \equiv 0 \pmod{m}$, $\quad 0 \le x \le m-1$, or

(II) $j + bx + (3m-2-x)a \equiv 0 \pmod{m}$, $\quad m-1 \le x \le 2m-1$.

But this implies $x(b-a) = 2a - j$ (in Z_m) and because $\gcd(a-b, m) = 1$, $b-a$ is a unit in Z_m so $x = (2a-j)(b-a)^{-1} \in Z_m$ is a solution. Hence in case (I) just take $x = (2a-j)(b-a)^{-1}$ and in case (II) take $x_0 = x + m \in [m-1, \ldots, 2m-1]$. Hence $\mathrm{ZS}(Z_3 \oplus Z_{3m}) = 12m + 1$.

The proof of the two other cases is exactly the same. ∎

Theorem 4 suggest the following conjecture:

CONJECTURE 2:

$$\mathrm{ZS}(Z_n \oplus Z_{nm}) = n(n+1)m + n - 2.$$

The only obstacles are that we depend in the former proof on the $4n-3$ bound for the $Z_n \oplus Z_n$ conjecture and that the Baker–Schmidt holds only for prime power. Anyway, this conjecture holds true for $n = 2^\alpha, 3^\alpha, 5^\alpha$ because in these cases the Baker–Schmidt theorem applies and the $4n-3$ bound for $Z_n \oplus Z_n$ is true by a multiplicative argument presented in [3]. It is also known that Conjecture 0 implies Conjecture 2 in view of a theorem of Olson [26] concerning $D(Z_n \oplus Z_{nm})$.

An important remark, after Theorem 4, that will be useful later is:

Remark: Suppose $G = Z_3 \oplus Z_{3n} \oplus H$, where $|H| = m$ and $\gcd(n, m) > 1$. Then $\mathrm{ZS}(G) \le 12nm < \frac{4|G|}{3} + 1$. ∎

Indeed we may repeat the proof of Theorem 4, step by step, ensuring $4nm-2$ triples, $A_1, \dots, A_{4nm-2}$, $|A_i| = 3$, such that each triple sums to 0 in $Z_3 \oplus Z_3$. Defining the d_i's as before, and since we are left with $Z_n \oplus H$ which is not cyclic as $\gcd(n,m) > 1$, and also $|Z_n \oplus H| = nm$, we can apply (after Theorem C), Case 1 in the proof of Theorem 4 to ensure $I \subset \{1, \dots, 4nm-2\}$, $|I| = 3nm$, $\sum_{i\in I} d_i = 0$ in $Z_n \oplus H$. Hence

$$\sum_{\substack{g_j \in A_i \\ i \in I}} g_j = 0 \quad \text{in } Z_3 \oplus Z_{3n} \oplus H.$$

4. The presence of abelian non-cyclic p-subgroups, $p \geq 5$

Our main goal in this section is to show that the presence of an abelian non-cyclic p-subgroup, $p \geq 5$, implies the inequality $\mathrm{ZS}(G) < \frac{4|G|}{3} + 1$. This goal is achieved through a sequence of computational propositions.

PROPOSITION 1: *Let $G = \bigoplus_{i=1}^{k} Z_{p^{e_i}} \oplus H$ where $e_i \geq t_i$. Then*

$$\mathrm{ZS}(G) \leq \left(1 + \frac{\left(\sum\limits_{i=1}^{k} p^{t_i}\right) - k + 1}{p^{t_1+t_2+\cdots+t_k}}\right)|G| - 1.$$

Proof: Set $G = A \oplus H$, then by Theorem 3, $\mathrm{ZS}(G) \leq (\mathrm{ZS}(A)-1)|H| + \mathrm{ZS}(H)$. Rearranging and using $\mathrm{ZS}(H) \leq 2|H|-1$ we obtain $\mathrm{ZS}(G) \leq (\mathrm{ZS}(A)+1)|H|-1$. Set $S = \bigoplus_{i=1}^{k} Z_{p^{e_i}}$ and apply Theorem 2 to get

$$(\mathrm{ZS}(A)+1)|H| - 1 = (|A| + D(A))|H| - 1 = \left(1 + \frac{D(A)}{|A|}\right)|G| - 1.$$

Applying monotonicity we are done. ∎

PROPOSITION 2:

(1) *Suppose $G = Z_{p^\alpha} \oplus Z_{p^\beta} \oplus H$ for some $p \geq 7$, $\alpha, \beta \geq 1$. Then $\mathrm{ZS}(G) < \frac{62}{49}|G| < \frac{4|G|}{3} + 1$.*

(2) *Suppose $G = Z_{5^\alpha} \oplus Z_{5^\beta} \oplus H$, $\alpha, \beta \geq 2$. Then $\mathrm{ZS}(G) < \frac{134}{125}|G| < \frac{4|G|}{3} + 1$.*

(3) *Suppose $G = Z_5 \oplus Z_{5m} \oplus H$, $m \geq 2$. Then $\mathrm{ZS}(G) < \frac{4|G|}{3} + 1$.*

(4) *Suppose $G = Z_5 \oplus Z_5 \oplus H$. Then $\mathrm{ZS}(G) < \frac{4|G|}{3} + 1$.*

Proof: Cases (1) and (2) follow directly from Proposition 1.

Case (3) follows from Theorem 4 if H is trivial, and otherwise from Theorem 4 and Proposition 1 since $m \geq 2$.

Case (4) follows from the observation that if H contains Z_{p^α}, $p \neq 5$, then $G = Z_5 \oplus Z_{5p^\alpha} \oplus H'$ and this case is solved in (3). Otherwise

$$G = Z_5 \oplus Z_5 \oplus Z_{5^\alpha} \oplus H'$$

and we are done by Proposition 1. ∎

The next theorem summarizes the content of Section 4.

THEOREM 5: *Suppose G contains an abelian non-cyclic p-group for some $p \geq 5$. Then* $\mathrm{ZS}(G) < \frac{4|G|}{3} + 1$.

Proof: It follows from Propositions 1 and 2 and Theorem 4. ∎

5. The presence of abelian non-cyclic p-subgroups, $p = 2, 3$

We now consider the presence of an abelian non-cyclic p-subgroup where $p = 2, 3$.

PROPOSITION 3:

(1) *Suppose $G = Z_{3^\alpha} \oplus Z_{3^\beta} \oplus Z_{3^\gamma} \oplus H$, $\alpha, \beta, \gamma \geq 1$. Then* $\mathrm{ZS}(G) < \frac{34}{27}|G| < \frac{4|G|}{3} + 1$.

(2) *Suppose $G = Z_{3^\alpha} \oplus Z_{3^\beta} \oplus H$, $\beta \geq \alpha \geq 2$. Then* $\mathrm{ZS}(G) < \frac{98}{81}|G| < \frac{4|G|}{3} + 1$.

Proof: Both cases follow directly from Proposition 1. ∎

Now we are left with the case $G = Z_3 \oplus Z_{3^\alpha} \oplus H$, where H contains no 3-subgroup. If H is cyclic, say $H = Z_n$, then $G = Z_3 \oplus Z_{3m}$, $m = 3^{\alpha-1}n$, and we proved in Theorem 4 that $\mathrm{ZS}(Z_3 \oplus Z_{3m}) = \frac{4|G|}{3} + 1$. If H is not cyclic we can write $H = Z_n \oplus H'$, hence $G = Z_3 \oplus Z_{3m} \oplus H'$, $m = 3^{\alpha-1}n$ and, by the remark after Theorem 4, $\mathrm{ZS}(G) < \frac{4|G|}{3} + 1$, hence we have proved:

THEOREM 6: *Suppose G contains an abelian non-cyclic 3-subgroup. Then*

$$\mathrm{ZS}(G) \leq \frac{4|G|}{3} + 1;$$

equality holds iff $G = Z_3 \oplus Z_{3n}$.

PROPOSITION 4:

(1) *Suppose $G = Z_{2^\alpha} \oplus Z_{2^\beta} \oplus Z_{2^\gamma} \oplus Z_{2^\delta} \oplus H$, $\alpha, \beta, \gamma, \delta \geq 1$. Then* $\mathrm{ZS}(G) < \frac{21}{16}|G| < \frac{4|G|}{3} + 1$.

(2) *Suppose $G = Z_{2^\alpha} \oplus Z_{2^\beta} \oplus Z_{2^\gamma} \oplus H$, $\alpha \geq 1$, $\beta, \gamma \geq 2$. Then* $\mathrm{ZS}(G) < \frac{40}{32}|G| < \frac{4|G|}{3} + 1$.

(3) *Suppose* $G = Z_2 \oplus Z_2 \oplus Z_{2^\alpha} \oplus H$ *where* H *is non-cyclic. Then* $\mathrm{ZS}(G) < \frac{4|G|}{3} + 1$.

(4) *Suppose* $G = Z_{2^\alpha} \oplus Z_{2^\beta} \oplus H$, $\beta \geq \alpha \geq 3$. *Then* $\mathrm{ZS}(G) < \frac{79}{64}|G| < \frac{4|G|}{3} + 1$.

Proof: Cases (1), (2) and (4) follow directly from Proposition 1. For case (3) we infer that since H is not cyclic, H must contain either a 2-subgroup and we are done by Proposition 4 (1), or a non-cyclic 3-subgroup and we are done by Theorem 6, or a non-cyclic p-subgroup for some $p \geq 5$ and we are done by Theorem 5.

PROPOSITION 5: *Suppose* $G = Z_4 \oplus Z_{2^\alpha} \oplus H$, $\alpha \geq 2$. *Then* $\mathrm{ZS}(G) < \frac{4|G|}{3} + 1$.

Proof: If H is cyclic of odd order then $G = Z_4 \oplus Z_{4n}$, $n = |H| \cdot 2^{\alpha-2}$ and, again by Theorem 4, $\mathrm{ZS}(G) < \frac{4|G|}{3} + 1$.

If H contains a 2-subgroup, then we are done by Proposition 4 (2) since then $\mathrm{ZS}(G) < \frac{40}{32}|G|$. Lastly, if H is non-cyclic then it contains a non-cyclic p-subgroup. If $p = 2$ we are done by Proposition 4. If $p = 3$ we are done by Theorem 6, and if $p \geq 5$ we are done by Theorem 5. ∎

So there remain to consider only the following three cases:

(1) $G = Z_2 \oplus Z_2 \oplus H$.

(2) $G = Z_2 \oplus Z_2 \oplus Z_{2^\alpha} \oplus H$, H is cyclic.

(3) $G = Z_2 \oplus Z_4 \oplus H$.

However, all these cases reduced to either $Z_2 \oplus Z_{2n}$, which is solved in Theorem 4, or to $Z_2 \oplus Z_2 \oplus Z_2 \oplus H$, where H is cyclic of odd order.

Indeed if $G = Z_2 \oplus Z_2 \oplus H$ and H is cyclic of order n, then either $G = Z_2 \oplus Z_{2n}$ if n is odd or $G = Z_2 \oplus Z_2 \oplus Z_{2^\alpha} \oplus H'$ where H' is cyclic of odd order. If H is not cyclic, then H contains an abelian non-cyclic p-subgroup and we are done by Proposition 4, Theorem 5, and Theorem 6.

If $G = Z_2 \oplus Z_4 \oplus H$ and H is cyclic of order n, then either $G = Z_2 \oplus Z_{2m}$, $m = 2n$ if n is odd, or $G = Z_2 \oplus Z_4 \oplus Z_{2^\alpha} \oplus H'$ where H is cyclic of odd order and in fact $\alpha = 1$ by Proposition 4. If H is not cyclic, then as before we are done by either Proposition 4, Theorem 5 or Theorem 6. So we are left with $G = Z_2 \oplus Z_2 \oplus Z_{2^\alpha} \oplus H$ where H is cyclic of odd order, by Proposition 4.

Our last result which completes the proof of Theorem 1 is:

PROPOSITION 6: *Suppose* $G = Z_2 \oplus Z_2 \oplus Z_{2n}$, *then* $\mathrm{ZS}(G) < \frac{4|G|}{3} + 1$.

Proof: We shall modify the proof of Theorem 4, to show first that $\frac{5|G|}{4}+3$ is an upper bound.

Let $A=\{a_1,\ldots,a_{10n+3}\}$ be a sequence of elements in G. Consider first the elements over $Z_2\oplus Z_2\oplus Z_2=H$. Every 9 elements of H contains two equal elements whose sum is 0. Also, by a routine application of the Baker-Schmidt theorem every 5 members contains either 2 of 4 elements whose sum is 0 in H. Hence in $10n+3$ members we have either $5n-1$ pairs, $A_1,\ldots,A_{5n-1}$, $|A_i|=2$, whose sum is 0 in H, or $5n-2$ such pairs and a quadruple B, $|B|=4$ whose sum is 0 in H, and we can apply the technique of the proof of Theorem 4 to obtain the desired result, namely that $\mathrm{ZS}(G)\le\frac{5|G|}{4}+3$.

Observe now that $\frac{5|G|}{4}+3<\frac{4|G|}{3}+1$ holds for $|G|>24$, hence for $n>3$. Since $\mathrm{ZS}(Z_2\oplus Z_2\oplus Z_2)=11<\frac{4\cdot 8}{3}+1$ and $\mathrm{ZS}(Z_2\oplus Z_2\oplus Z_4)=21<\frac{4\cdot 16}{3}+1$, we are left only with $G=Z_2\oplus Z_2\oplus Z_6$, for which we have to show $\mathrm{ZS}(G)\le 32$. Here is the ad-hoc computation. Let $A=\{a_1,\ldots,a_{32}\}$ be a sequence of elements in $G=Z_2\oplus Z_2\oplus Z_2\oplus Z_3$. By the arguments above one of the following cases occurs.

CASE 1: There are 14 $(5n-1)$ pairs whose sum is 0 in $Z_2\oplus Z_2\oplus Z_2$ and we can apply the techniques of Theorem 4.

CASE 2: There are 13 $(5n-2)$ pairs whose sum is 0 in $Z_2\oplus Z_2\oplus Z_2$ and a quadruple whose sum is 0 in $Z_2\oplus Z_2\oplus Z_2$ and again we can apply the techniques of Theorem 4.

CASE 3: There are 12 $(5n-3)$ pairs whose sum is 0 and we are left with 8 elements, no two of them equal over $Z_2\oplus Z_2\oplus Z_2$ (otherwise we are in case 1 or 2). However, in this case the 8 elements form the whole group $Z_2\oplus Z_2\oplus Z_2$ and we can write them as follows:

$$g_1=(0,0,0,b_1),\quad g_2=(0,0,1,b_2),\quad g_3=(0,1,0,b_3),\quad g_4=(1,0,0,b_4),$$
$$g_5=(0,1,1,b_5),\quad g_6=(1,0,1,b_6),\quad g_7=(1,1,0,b_7),\quad g_8=(1,1,1,b_8),$$

where b_i is the Z_3-component. By Theorem A we can choose 9 of the 12 pairs so that their sum is 0 in $Z_2\oplus Z_2\oplus Z_2\oplus Z_3$. This forms a subsequence of 18 elements whose sum is 0 in G. We are left with three pairs A_1, A_2, A_3, $|A_i|=2$, $i=1,2,3$ whose sum is 0 in $Z_2\oplus Z_2\oplus Z_2$ (each) and denote by C_1, C_2, C_3 respectively their sum in the fourth coordinate (the Z_3-coordinate).

If $C_1 = C_2 = C_3$ or $C_1 \neq C_2 \neq C_3 \neq C_1$ then we may add A_1, A_2, A_3 to the former 18 elements to get 24 elements whose sum is 0 in G. Hence without loss of generality $C_1 \neq C_2 = C_3$. Now observe that $\{g_1, g_2, \ldots, g_8\}$ contains many subsequences of 4 elements whose sum is 0 in $Z_2 \oplus Z_2 \oplus Z_2$ and if there exists two such quadruples, say B_1, B_2, with distinct sum (mod 3) in the last coordinate, then there must exist $1 \leq i \leq 3$, $i \leq j \leq 2$ such that $A_i \cup B_j$ is a 6-tuple with zero-sum in G which we can add to the former 18 elements and we are done. So consider: $B_1 = \{g_4, g_3, g_7, g_1\}$, $B_2 = \{g_4, g_2, g_6, g_1\}$, $B_3 = \{g_4, g_8, g_5, g_1\}$. These are quadruples whose sum is 0 in $Z_2 \oplus Z_2 \oplus Z_2$ and hence their sum in the Z_3-coordinate must be equal. But this sum is $b_4 + b_3 + b_7 + b_1 \equiv b_4 + b_2 + b_6 + b_1 \equiv b_4 + b_8 + b_5 + b_1 \equiv j \pmod 3$. Hence summing over all of them we get:

$$3b_4 + 3b_1 + b_2 + b_3 + b_4 + b_5 + b_6 + b_7 \equiv 0 \pmod 3$$

hence

$$b_2 + b_3 + b_4 + b_5 + b_6 + b_7 \equiv 0 \pmod 3,$$

but also it is easy to check now that:

$$g_2 + g_3 + g_4 + g_5 + g_6 + g_7 = 0 \text{ in } Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_3$$

and again we can add these elements to the former 18 elements to obtain 24 members of A whose sum is 0 in $Z_2 \oplus Z_2 \oplus Z_6$.

This completes the proof of Proposition 6 and the main theorem of this paper. ∎

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